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Electrostatic repulsion of charged pith balls hanging from strings

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Abstract
Two positively charged pith balls hang from a nail at the end of equal-length strings in Earth's surface gravitational field. The problem consists in finding each of the hanging angles when the balls do not necessarily have the same mass or charge. The solution is an excellent exercise in developing two skills: wisely choosing the coordinate axes in a free-body diagram, and correctly interpreting the roots and limits of a numerical solution. The treatment is accessible to undergraduate physics majors in their first or second year of physics courses.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

A standard demonstration illustrating Coulomb's law consists of a pair of identically charged pith balls hung by lightweight strings from a common point of attachment [1, 2]. As an exercise in working with free-body diagrams, many textbooks [3, 4] discuss the problem of finding the charge on the balls, given the angle the strings make with the vertical. This paper considers the inverse problem of predicting the hanging angles for known charges. This apparently simple variation leads to a considerably more complicated solution. If the two balls have the same mass, an analytic solution can be obtained using the cubic equation. However, numerical calculations are necessary if their masses are different.

For generality, assume one ball has mass M_1 and positive charge Q_1, and the other mass M_2 and positive charge Q_2. (The spheres are small enough that they can be treated as point charges.) Both strings have the same length L. These five values are assumed to be given. The problem consists in finding the angles θ_1 and θ_2 at which the two strings hang relative to the vertical, as illustrated in figure 1.

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14. ABSTRACT
Two positively charged pith balls hang from a nail at the end of equal-length strings in Earth?s surface gravitational field. The problem consists in finding each of the hanging angles when the balls do not necessarily have the samemass or charge. The solution is an excellent exercise in developing two skills: wisely choosing the coordinate axes in a free-body diagram, and correctly interpreting the roots and limits of a numerical solution. The treatment is accessible to undergraduate physics majors in their first or second year of physics courses.

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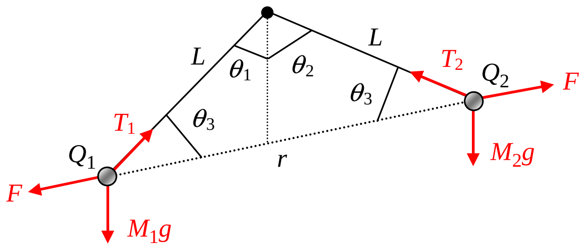

Figure 1. Two charged pith balls hanging on massless strings from a nail. The tensions in the two strings are T_1 and T_2.

2. Equations determining the general solution

Denoting the distance between the two balls as r, the triangle in figure 1 bounded by that distance and the two strings is isosceles, and hence the two angles labelled θ_3 are equal. Consequently

$$r = 2L\cos\theta_3 \tag{1}$$

so that the electrostatic force of repulsion between the two balls has magnitude

$$F = \frac{kQ_1Q_2}{r^2} = \frac{kQ_1Q_2}{4L^2\cos^2\theta_3}, \tag{2}$$

where k is the Coulomb constant. The two charges appear in the problem only in this combination, and thus it is not their individual values that affect the angles but only their product. Accordingly, it makes sense to replace their product by the square of their geometric mean, $Q^2 \equiv Q_1Q_2$.

Noting that the balls are in static equilibrium, the three forces on each must sum to zero. Therefore the components of the forces on a ball perpendicular to its suspending string must balance. For ball 1, that balancing relation is

$$M_1g\sin\theta_1 = F\sin\theta_3, \tag{3}$$

and likewise for ball 2,

$$M_2g\sin\theta_2 = F\sin\theta_3, \tag{4}$$

where g is the gravitational field strength. Because the right-hand sides of equations (3) and (4) are equal, their left-hand sides must also be equal, so that

$$M_1g\sin\theta_1 = M_2g\sin\theta_2 \quad \Rightarrow \quad \sin\theta_2 = m\sin\theta_1, \tag{5}$$

where $m \equiv M_1/M_2$ is the (dimensionless) ratio of the masses of the two balls. Consequently if one of the two hanging angles θ_1 or θ_2 is known, then the other can be immediately calculated.[1] The problem thus reduces to finding one of the two angles, say θ_1. For definiteness, assume that if one ball is heavier than the other, it is labelled as ball 1, i.e. $m \geqslant 1$. Then that ball can never reach the horizontal position (i.e. $0 \leqslant \theta_1 < \pi/2$), but the second ball can rotate around as far as the vertical for appropriate charges and masses (i.e. $0 \leqslant \theta_2 \leqslant \pi$).

Returning to the isosceles triangle in figure 1, the sum of its interior angles must be π:

$$\theta_1 + \theta_2 + 2\theta_3 = \pi \quad \Rightarrow \quad \theta_3 = \frac{\pi}{2} - \left(\frac{\theta_1 + \theta_2}{2}\right). \tag{6}$$

[1] It is difficult to derive equation (5) if one adopts standard horizontal–vertical axes or if one insists on using the same set of coordinate axes for both spheres. This feature makes it a good example problem to use in class!

Substitute equation (2) into (3) and then use equation (6), noting that cosine of an angle equals sine of its complementary angle and vice versa, to obtain

$$\sin\theta_1 = \frac{f}{4}\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin^{-2}\left(\frac{\theta_1+\theta_2}{2}\right), \tag{7}$$

where $f \equiv kQ^2/M_1 g L^2$ is a (dimensionless) ratio of forces. Next the half-angle and double-angle formulae for sine and cosine can be employed to rewrite equation (7) as

$$\sin\theta_1 = \frac{f}{2\sqrt{2}}\frac{\sqrt{1+\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2}}{1-\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2}. \tag{8}$$

Finally equation (5) can be used, together with the Pythagorean identity $\cos^2\theta = 1-\sin^2\theta$, to re-express equation (8) as

$$s = \frac{f}{2\sqrt{2}}\frac{\sqrt{1+C\sqrt{1-s^2}\sqrt{1-m^2s^2}-ms^2}}{1-C\sqrt{1-s^2}\sqrt{1-m^2s^2}+ms^2}, \tag{9}$$

where s is a shorthand for $\sin\theta_1$ and C is the sign of $\cos\theta_2$, i.e. $C = +1$ if $0 < \theta_2 < \pi/2$, and $C = -1$ if $\pi/2 < \theta_2 < \pi$. Assuming that sign can be figured out, equation (9) in principle completely determines the value of θ_1, since m and f are dimensionless constants that can be calculated from the givens. Finally equation (5) can then be used to compute the value of θ_2, where C determines whether the solution of the inverse sine function, $\sin^{-1}(ms)$, should be in the first or second quadrant.

3. Analytic solution for the special case of equal-mass pith balls

If the two balls have equal mass M, then $m = 1$ and $\theta_1 = \theta_2 \equiv \theta$ (even if the balls do not have equal charges). In that case, $C = +1$. As the mean charge Q increases, the balls increasingly repel and the angles rise from 0 towards $\pi/2$. But the strings can never reach (or surpass) the horizontal position because there would then be no upward component of the tension to balance each ball's weight (noting that the electrostatic force F is purely horizontal for equal masses). Squaring equation (9) and rearranging it leads to the cubic equation

$$16x^3 + f^2x - f^2 = 0, \tag{10}$$

where $x \equiv \sin^2\theta$. Cardano's formula then gives the unique real solution:

$$\sin^2\theta = \frac{f^{2/3}}{2^{5/3}}(1+\sqrt{1+f^2/108})^{1/3} - \frac{f^{4/3}}{12\cdot 2^{1/3}}(1+\sqrt{1+f^2/108})^{-1/3}. \tag{11}$$

This result for θ is plotted versus $f \equiv kQ^2/MgL^2$ in figure 2. As expected, the angle increases as the charge on either sphere increases or as the mass decreases. When $f = 2$ the angle is exactly $\theta = \pi/4$, as can be verified easily from equation (7).

4. Numerical solution for unequal-mass balls

For any value of $m > 1$, there exists a value of the mean charge (and hence of f) for which $\theta_2 = \pi/2$. At that angle, $\cos\theta_2 = 0$ and $\sin\theta_2 = 1$, and hence equation (5) implies that $\sin\theta_1 = 1/m$. Inserting these values into equation (8) leads to a critical value of f of

$$f_c = \frac{(2\sqrt{2}/m)(1+1/m)}{\sqrt{1-1/m}}. \tag{12}$$

(In agreement with figure 2, this equation implies that for $m = 1$ the hanging angle can only attain $\pi/2$ when $f \to \infty$.) If $f < f_c$ then $C = +1$ in equation (9), whereas if $f > f_c$ then

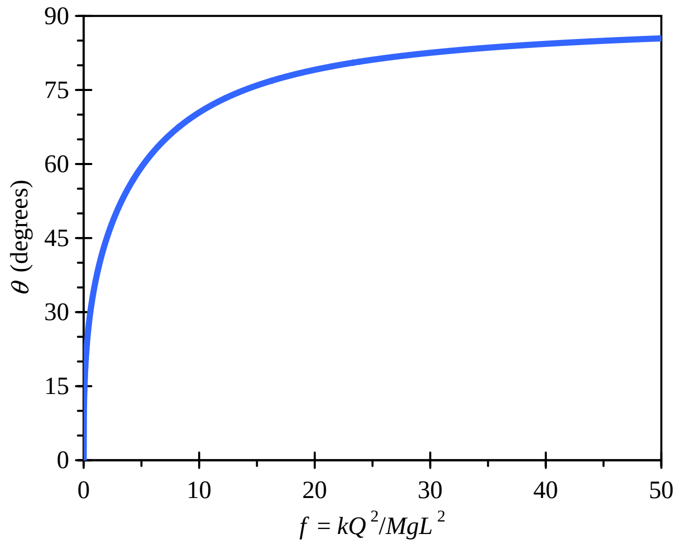

Figure 2. Half-angle between the strings for the equal-mass case. The abscissa quantifies a dimensionless ratio of the product of the charges to the mass of either ball.

$C = -1$, for any given value of the mass ratio m. Consider what happens if Q were to increase in value starting from zero for fixed masses of the two balls. Initially both θ_1 and θ_2 would increase from zero, but only until $f = f_c$. At that point, string 2 will be horizontal and so $\sin\theta_2$ will have attained its maximum value of 1. As f further increases, $\sin\theta_2$ must decrease. But then equation (5) implies that $\sin\theta_1$ must decrease, although $\theta_1 < \pi/2$. That necessarily means θ_1 must decrease. However, an increase in f must cause the separation distance between the two balls to increase, owing to the stronger electrostatic repulsion. Consequently, θ_2 must increase (beyond $\pi/2$) by more than θ_1 decreases.

As a specific example, suppose that ball 1 is twice as heavy as ball 2, so that $m = 2$. Then equation (12) becomes $f_c = 3$. Equation (9) was numerically solved[2] to obtain θ_1 for values of f starting from zero and increasing in steps of 0.02, using $C = +1$ for $f < 3$ and $C = -1$ for $f > 3$. The result is plotted as the lower curve in figure 3. Then θ_2 was computed using equation (5) to give the upper curve in that figure. Angle θ_1 increases from 0 to $\pi/6$, and θ_2 increases from 0 to $\pi/2$, as f increases from 0 to 3. Beyond $f = 3$, θ_1 decreases back to 0, while θ_2 rises to π. In contrast to figure 2, however, these limiting angles are not reached asymptotically, but at a definite value $f_{\max}$. In particular, when $C = -1$ the right-hand side of equation (9) expanded to lowest nonzero order in s is equal to $fs\,(m-1)/8$, and thus[3] $f_{\max} = 8/(m-1)$. When $m = 2$, this result implies that $\theta_1 \to 0$ as $f \to 8$, in agreement with figure 3. If f is increased beyond $f_{\max}$, then ball 1 becomes more firmly pinned at $\theta_1 = 0$ and ball 2 at $\theta_2 = \pi$ as the tensions in the two strings rise.

When $\theta_1 = 0$ and $\theta_2 = \pi$ (so that $r = 2L$), the tension in string 2 will just fall to zero when

$$F = M_2 g \quad \Rightarrow \quad f_{\text{slack}} = \frac{4}{m}. \tag{13}$$

[2] The command ‘Solve’ was used in Mathematica™ for this purpose, but any root finder or equation solver on a programmable calculator or in a mathematical software package should be able to do the job.

[3] Note that $f_{\max} > f_c$ for any $m > 1$.

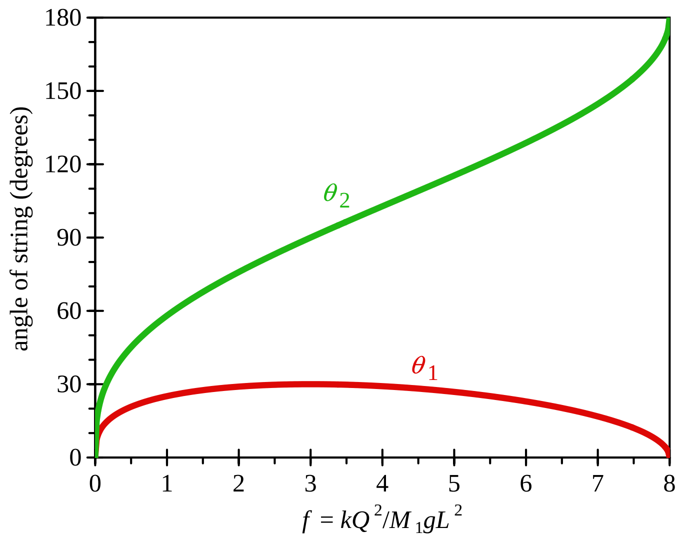

Figure 3. Hanging angles when the first ball is twice as massive as the second ball.

Since that value is smaller than $f_{\max}$ for any $m > 1$, there is no danger of the string going slack. However, that is only true if the two balls both begin at zero angle when uncharged and move along circular arcs as they are increasingly charged up and repel one another. One might instead permit ball 1 to remain at $\theta_1 = 0$ and ball 2 to be repelled vertically straight upwards. In particular, note that equation (9) has a solution of $\theta_1 = 0$ when $C = -1$ for *any* value of f! For example, in the case of $m = 2$, there is a stable configuration (i.e. the upper string is taut) with $\theta_1 = 0$, $\theta_2 = \pi$, and $2 < f < 8$ that is inaccessible unless ball 2 is allowed to suddenly jump up to the top of the circle, rather than having to circle halfway around the perimeter. When working with equation (9), one therefore needs to be careful in selecting the solution corresponding to the desired physical situation and not just accept any output from a numerical root finder. That is a useful lesson for students to learn.

The solutions for the positions of the pith balls computed here (as plotted for particular cases in figures 2 and 3) are stable against small perturbations of the hanging angles within the plane of figure 1[4]. A straightforward way to demonstrate this fact is to show that the potential energy of the system is a minimum. The total potential energy U is the sum of the gravitational potential energy of each ball and the electrostatic potential energy of interaction between them,

$$U = M_1 g L(1 - \cos\theta_1) + M_2 g L(1 - \cos\theta_2) + kQ^2/2L \sin\left(\frac{\theta_1 + \theta_2}{2}\right), \qquad (14)$$

using equations (1) and (6), where the gravitational reference level is taken to be at the lowest point that either ball can hang (so that the system has $U = 0$ when the balls are uncharged). Equation (14) can be differentiated with respect to θ_1 using the fact that $\mathrm{d}\theta_2/\mathrm{d}\theta_1 = M_1 \cos\theta_1 / M_2 \cos\theta_2$ from equation (5). If equation (7) is substituted into the resulting first derivative, one finds $\mathrm{d}U/\mathrm{d}\theta_1 = 0$ consistent with the fact that the forces balance at the angles described by equation (7). With a bit more work, one can compute the second derivative of equation (14) and again insert equation (7) into the result to verify

[4] Rotations about a vertical axis of the plane of the balls and strings can be avoided if the nail in figure 1 is banged into a wall rather than into the ceiling.

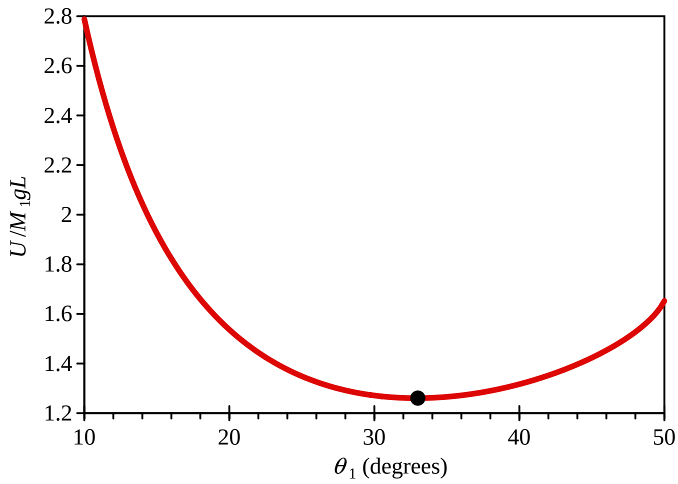

Figure 4. Normalized potential energy of the system plotted as a function of the hanging angle of the heavier pith ball for the example of $f \equiv kQ^2/M_1gL^2 = 1.1$ and $m \equiv M_1/M_2 = 1.3$. The black dot is at the angle obtained from a numerical solution of equation (9) for these values of f and m, noting that $C = +1$ according to equation (12), thereby demonstrating that the potential energy is a true minimum and thus that this numerical solution is stable.

that $d^2U/d\theta_1^2 > 0$ for any allowed values of the masses and angles, thereby proving that the solutions are stable. Rather than slogging through all that differentiation and algebra, a simpler approach is to simply plot equation (14), normalized by M_1gL so that it depends only on the two parameters f and m, as a function of θ_1 where $\theta_2 = \sin^{-1}(m \sin\theta_1)$ according to equation (5). An example is shown in figure 4 for the case of $f = 1.1$ and $m = 1.3$. A minimum is observed at the black dot in the figure, in agreement with the numerical value of $\theta_1 = 32.89°$ obtained by finding the root of equation (9). This graphical method of solution is thus an alternative to deriving and solving that latter equation.

Readers interested in extending the work presented here are invited to have their students plot the hanging angles as a function of the mass ratio for fixed mean charge. Another possible project would be to experimentally confirm figure 3 by delivering known charges to foil-wrapped pith balls.

Acknowledgments

This paper was motivated by a discussion on the PHYS-L list entitled 'Electrostatics problem' that can be searched for at https://carnot.physics.buffalo.edu/archives/. Part of it forms the basis of an upcoming Physics Challenge to be published in *The Physics Teacher*.

References

[1] O'Leary A J 1947 Definition of electric charge derived from simple quantitative experiments with pith balls *Am. J. Phys.* **15** 417–21
[2] Wiley P H and Stutzman W L 1978 A simple experiment to demonstrate Coulomb's law *Am. J. Phys.* **46** 1131–2
[3] Serway R A and Jewett J W 2010 *Physics for Scientists and Engineers* 8th edn (Belmont, CA: Brooks /Cole) p 666
[4] Halliday D, Resnick R and Walker J 2011 *Fundamentals of Physics* 9th edn (Hoboken, NJ: Wiley) p 578